Solution to HW8

Leon Li

ylli@math.cuhk.edu.hk

MATH 2020 B   HW 8

Due Date: Apr 17, 2020 (12:00 noon)

Thomas' Calculus (12th Ed.)

§16.4: 6, 9, 13, 15, 22

24, 29, 30, 34, 39

§16.4

Circulation and Flux

In Exercises 5–14, use Green's Theorem to find the counterclockwise circulation and outward flux for the field **F** and curve *C*.

6. $\mathbf{F} = (x^2 + 4y)\mathbf{i} + (x + y^2)\mathbf{j}$

C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

Sol) Let $\vec{F}(x,y) = (x^2+4y)\vec{i} + (x+y^2)\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

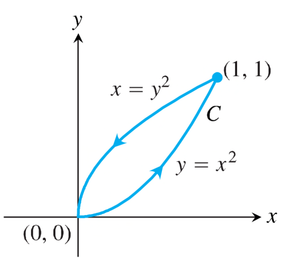
Then $M_x = 2x$; $M_y = 4$; $N_x = 1$; $N_y = 2y$.

The region bounded by C is given by $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \ ; \ 0 \le y \le 1\}$

$\therefore$ By Green's Thm, Circulation $= \iint_D (N_x - M_y)\,dA = \int_0^1\int_0^1 (1-4)\,dy\,dx = -3$ //

Flux $= \iint_D (M_x + N_y)\,dA = \int_0^1\int_0^1 (2x+2y)\,dy\,dx = \int_0^1 [2xy + y^2]_0^1\,dx = \int_0^1 (2x+1)\,dx = 2$ //

9. $\mathbf{F} = (xy + y^2)\mathbf{i} + (x - y)\mathbf{j}$



Sol) Let $\vec{F}(x,y) = (xy+y^2)\vec{i} + (x-y)\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$

Then $M_x = y$; $M_y = x + 2y$; $N_x = 1$; $N_y = -1$.

The region bounded by C is given by

$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \ ; \ x^2 \le y \le \sqrt{x}\}$

$\therefore$ By Green's Thm, Circulation $= \iint_D (N_x - M_y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} (1-(x+2y))\,dy\,dx$

$= \int_0^1 [(1-x)y - y^2]_{x^2}^{\sqrt{x}}\,dx = \int_0^1 \left((1-x)(\sqrt{x}-x^2) - (x-x^4)\right)dx = \int_0^1 (\sqrt{x} - x - x\sqrt{x} - x^2 + x^3 + x^4)\,dx$

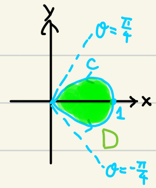
$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} - \frac{x^{5/2}}{5/2} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = \left(\frac{2}{3} - \frac{1}{2} - \frac{2}{5} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = -\frac{7}{60}$.

Flux $= \iint_D (M_x + N_y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} (y-1)\,dy\,dx = \int_0^1 \left[\frac{y^2}{2} - y\right]_{x^2}^{\sqrt{x}}\,dx = \int_0^1 \left(\frac{x}{2} - \frac{x^4}{2} - \sqrt{x} + x^2\right)dx$

$= \left[\frac{x^2}{4} - \frac{x^5}{10} - \frac{x^{3/2}}{3/2} + \frac{x^3}{3}\right]_0^1 = \left(\frac{1}{4} - \frac{1}{10} - \frac{2}{3} + \frac{1}{3}\right) = -\frac{11}{60}$ //

13. $\mathbf{F} = (x + e^x \sin y)\mathbf{i} + (x + e^x \cos y)\mathbf{j}$

C: The right-hand loop of the lemniscate $r^2 = \cos 2\theta$



Sol) Let $\vec{F}(x,y) = (x + e^x \sin y)\vec{i} + (x + e^x \cos y)\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

Then $M_x = 1 + e^x \sin y$; $M_y = e^x \cos y$; $N_x = 1 + e^x \cos y$; $N_y = -e^x \sin y$.

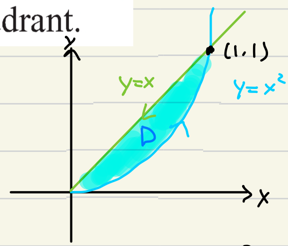
The region bounded by C is given by $D = \{(r,\theta) \mid -\frac{\pi}{4} \le \theta \le \frac{\pi}{4},\ 0 \le r \le \sqrt{\cos 2\theta}\}$

$\therefore$ By Green's Thm, Circulation $= \iint_D (N_x - M_y)\,dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} (1)(r\,dr\,d\theta)$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{\sqrt{\cos 2\theta}} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos 2\theta}{2}\,d\theta = \left[\frac{\sin 2\theta}{4}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2}$ //

Flux $= \iint_D (M_x + N_y)\,dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} (1)(r\,dr\,d\theta) = \frac{1}{2}$ //

15. Find the counterclockwise circulation and outward flux of the field $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$ around and over the boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.



Sol) Let $\vec{F}(x,y) = xy\,\vec{i} + y^2\,\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

Then $M_x = y$; $M_y = x$; $N_x = 0$; $N_y = 2y$.

The region bounded by C is given by $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 ;\ x^2 \le y \le x\}$

$\therefore$ By Green's Thm, Circulation $= \iint_D (N_x - M_y)\,dA = \int_0^1 \int_{x^2}^{x} (0 - x)\,dy\,dx$

$= -\int_0^1 x(x - x^2)\,dx = -\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = -\frac{1}{12}$ //

Flux $= \iint_D (M_x + N_y)\,dA = \int_0^1 \int_{x^2}^{x} (y + 2y)\,dy\,dx = \int_0^1 \left[\frac{3y^2}{2}\right]_{x^2}^{x} dx = \frac{3}{2}\int_0^1 (x^2 - x^4)\,dx$

$= \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{5}$ //

Using Green's Theorem

Apply Green's Theorem to evaluate the integrals in Exercises 21–24.

22. $\oint_C (3y\,dx + 2x\,dy)$

C: The boundary of $0 \le x \le \pi$, $0 \le y \le \sin x$

24. $\oint_C (2x + y^2)\,dx + (2xy + 3y)\,dy$

C: Any simple closed curve in the plane for which Green's Theorem holds

Sol) (22) Let $3y\vec{i} + 2x\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

Then $M_y = 3$; $N_x = 2$.

The region bounded by C is given by $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \pi ; 0 \le y \le \sin x\}$

$\therefore$ By Green's Thm, $\oint_C (M\,dx + N\,dy) = \iint_D (N_x - M_y)\,dA = \int_0^{\pi}\int_0^{\sin x} (2-3)\,dy\,dx$

$= \int_0^{\pi} -\sin x\,dx = [\cos x]_0^{\pi} = -2$

(24) Let $(2x+y^2)\vec{i} + (2xy+3y)\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

Then $M_y = 2y$; $N_x = 2y$.

The region bounded by C is given by D.

$\therefore$ By Green's Thm, $\oint_C (M\,dx + N\,dy) = \iint_D (N_x - M_y)\,dA = \iint_D 0\,dA = 0$

29. Let C be the boundary of a region on which Green's Theorem holds. Use Green's Theorem to calculate

a. $\oint_C f(x)\,dx + g(y)\,dy$

b. $\oint_C ky\,dx + hx\,dy$ (k and h constants).

Sol) (a) Let $f(x)\vec{i} + g(y)\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

Then $M_y = 0$; $N_x = 0$.

The region bounded by C is given by D.

$\therefore$ By Green's Thm, $\oint_C (M\,dx + N\,dy) = \iint_D (N_x - M_y)\,dA = \iint_D 0\,dA = 0$ //

(b) Let $ky\vec{i} + hx\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

Then $M_y = k$; $N_x = h$.

The region bounded by C is given by D.

$\therefore$ By Green's Thm, $\oint_C (M\,dx + N\,dy) = \iint_D (N_x - M_y)\,dA = \iint_D (h-k)\,dA = (h-k)\cdot \text{Area}(D)$ //

30. Integral dependent only on area Show that the value of

$$\oint_C xy^2\,dx + (x^2y + 2x)\,dy$$

around any square depends only on the area of the square and not on its location in the plane.

Sol) Let $xy^2\vec{i} + (x^2y+2x)\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

Then $M_y = 2xy$; $N_x = 2xy+2$. The region bounded by C is given by D.

$\therefore$ By Green's Thm, $\oint_C (M\,dx + N\,dy) = \iint_D (N_x - M_y)\,dA = \iint_D (2xy+2-2xy)\,dA = 2\cdot\text{Area}(D)$.

$\therefore$ $\oint_C (M\,dx + N\,dy)$ depends only on the area of D but not its location.

34. Definite integral as a line integral Suppose that a nonnegative function $y = f(x)$ has a continuous first derivative on $[a, b]$. Let C be the boundary of the region in the xy-plane that is bounded below by the x-axis, above by the graph of f, and on the sides by the lines $x = a$ and $x = b$. Show that

$$\int_a^b f(x)\,dx = -\oint_C y\,dx.$$

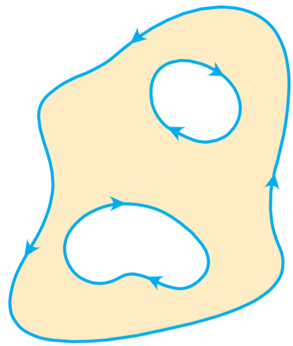
Sol) Let $-y\vec{i} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

Then $M_y = -1$; $N_x = 0$.

The region bounded by C is given by $D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b;\ 0 \le y \le f(x)\}$

$\therefore$ By Green's Thm, $-\oint_C y\,dx = \iint_D (N_x - M_y)\,dA = \int_a^b \int_0^{f(x)} 1\cdot dy\,dx = \int_a^b f(x)\,dx$

39. Regions with many holes Green's Theorem holds for a region R with any finite number of holes as long as the bounding curves are smooth, simple, and closed and we integrate over each component of the boundary in the direction that keeps R on our immediate left as we go along (see accompanying figure).



a. Let $f(x, y) = \ln(x^2 + y^2)$ and let C be the circle $x^2 + y^2 = a^2$. Evaluate the flux integral

$$\oint_C \nabla f \cdot \mathbf{n}\, ds.$$

Sol) (a) Let $\nabla f(x,y) = \frac{2x}{x^2+y^2}\vec{i} + \frac{2y}{x^2+y^2}\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

Case 1: C has anti-clockwise orientation Then C can be parametrized as

$\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$, where $0 \le t < 2\pi$.

Then $M(\vec{r}(t)) = \frac{2a\cos t}{a^2(\cos^2 t + \sin^2 t)} = \frac{2}{a}\cos t$; $N(\vec{r}(t)) = \frac{2a\sin t}{a^2(\cos^2 t+\sin^2 t)} = \frac{2}{a}\sin t$.

$\vec{r}\,'(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$; $\vec{n}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

$\therefore \oint_C \nabla f \cdot \vec{n}\, ds = \int_0^{2\pi} \left(\left(\frac{2}{a}\cos t\right)\cdot(a\cos t) + \left(\frac{2}{a}\sin t\right)\cdot(a\sin t)\right) dt = 2\int_0^{2\pi} dt = 4\pi$ //

Case 2: C has clockwise orientation Let C' be the circle with anticlockwise orientation.

Then $\oint_C \nabla f \cdot \vec{n}\, ds = -\oint_{C'} \nabla f \cdot \vec{n}\, ds = -4\pi$ //

Rmk The numerical answer 0, appeared in the textbook "Thomas' Calculus" (12 Edition) is incorrect.

b. Let K be an arbitrary smooth, simple closed curve in the plane that does not pass through $(0, 0)$. Use Green's Theorem to show that

$$\oint_K \nabla f \cdot \mathbf{n}\, ds$$

has two possible values, depending on whether $(0, 0)$ lies inside K or outside K.

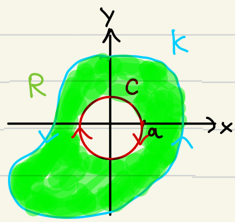
(b) Case 1: $(0,0)$ does not lie inside K. Let D be the region bounded by K.

Then by Green's Theorem, $\oint_K \nabla f \cdot \vec{n}\, ds = \pm \iint_D (M_x + N_y)\, dA$ (where ($\pm$) sign depends on the orientation of K)

$$= \pm \iint_D \left(\frac{(x^2+y^2)\cdot 2 - 2x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)\cdot 2 - 2y(2y)}{(x^2+y^2)^2} \right) dA = 0$$

Case 2: $(0,0)$ lies inside K. Choose $a > 0$ sufficiently small such that C lies inside K.

K has anticlockwise orientation and C has clockwise orientation.



Let R be the region lying between K and C.

Then since R does not enclose $(0,0)$, $0 = \iint_R (M_x + N_y)\, dA$ (by Case 1)

$= \oint_K \nabla f \cdot \vec{n}\, ds + \oint_C \nabla f \cdot \vec{n}\, ds$ (by Green's Theorem)

$= \oint_K \nabla f \cdot \vec{n}\, ds + (-4\pi)$ (by (a), case 2). $\therefore \oint_K \nabla f \cdot \vec{n}\, ds = 4\pi$.

Combining both cases, $\oint_K \nabla f \cdot \vec{n}\, ds = \begin{cases} 0, & \text{if } (0,0) \text{ does not lie inside } K. \\ 4\pi, & \text{if } (0,0) \text{ lies inside } K. \end{cases}$